ELSEVIER

Topology and its Applications 89 (1998) 179–202

TOPOLOGY AND ITS APPLICATIONS

Weighted colimits and formal balls in generalized metric spaces

J.J.M.M. Rutten [1]

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Received 30 January 1997; received in revised form 28 July 1997

Abstract

(a) Limits of Cauchy sequences in a (possibly nonsymmetric) metric space are shown to be weighted colimits (a notion introduced by Borceux and Kelly, 1975). As a consequence, further insights from enriched category theory are applicable to the theory of metric spaces, thus continuing Lawvere's (1973) approach. Many of the recently proposed definitions of generalized limit turn out to be theorems from enriched category theory.

(b) The dual of the space of metrical predicates ('fuzzy subsets') of a metric space is shown to contain the collection $\mathcal{F}$ of formal balls (Weihrauch and Schreiber, 1981; Edalat and Heckmann, 1996) as a quasi-metric subspace. Formal balls are related to ordinary closed balls by means of the Isbell conjugation. For an ordinary metric space X, the subspace of minimal elements of $\mathcal{F}$ is *isometric* to X by the co-Yoneda embedding. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Generalized metric spaces; Enriched categories; Weighted limits; Formal balls

AMS classification: 4A72; 18D20; 54E35; 68Q05

1. Motivation

(A) A Cauchy sequence in a metric space X is determined by the following data:

(1) a function $f:\{0,1,2,\ldots\}\to X$;

(2) a Cauchy condition: $\forall\varepsilon>0\ \exists N\geqslant 0\ \forall n\geqslant m\geqslant N,\ X(f(m),f(n))\leqslant\varepsilon$.

(Here $X(f(m),f(n))$ denotes the distance from $f(m)$ to $f(n)$.) The Cauchy condition is easily seen to be equivalent to the following:

(2′) There exists a function $g:\{0,1,2,\ldots\}\to[0,\infty]$ such that

(a) $\forall n\geqslant 0,\ g(n)\geqslant g(n+1)$;

[1] E-mail: janr@cwi.nl.

PII: S0166-8641(97)00224-1

(b) $\inf g(n) = 0$;

(c) $\forall n \geqslant 0\ \forall k \geqslant 0,\ X(f(n), f(n+k)) \leqslant g(n)$.

Let us call such a function g a *Cauchy witness* for the sequence f. (There are several such witnesses; a canonical choice would be the function g defined by

$$g(n) = \sup_{k \geqslant l \geqslant n} X\big(f(l), f(k)\big).)$$

A Cauchy witness gives for any natural number n the extent to which the sequence $(f(n+k))_k$ 'is Cauchy'. This functional description of the Cauchy condition gives rise to a useful alternative for the traditional definition of limit of a Cauchy sequence, which is repeated first: for $x \in X$,

$$(3)\ x = \lim f(n) \Leftrightarrow \forall \varepsilon > 0\ \exists N \geqslant 0\ \forall n \geqslant N,\ X\big(f(n), x\big) \leqslant \varepsilon.$$

Alternatively, the following definition is formulated in terms of f and a Cauchy witness g for f:

$$(3')\ x = \underset{\longrightarrow g}{\lim} f(n) \Leftrightarrow \forall y \in X,\ X(x,y) = \sup_{n \geqslant 0} \big\{X\big(f(n), y\big) \dot{-} g(n)\big\},$$

where $\dot{-} : [0,\infty] \times [0,\infty] \to [0,\infty]$ is truncated subtraction on the extended reals. As it turns out, the two definitions are equivalent:

Theorem 1.1. *Let f be a Cauchy sequence in a metric space X, and g a Cauchy witness for f. For all x in X,*

$$x = \lim f(n) \Leftrightarrow x = \underset{\longrightarrow g}{\lim} f(n).$$

(Note that as a consequence, definition (3′) *is independent of the choice of the witness g.)*

Proof. ($\Leftarrow$) It follows from

$$0 = X(x,x) = \sup_{n \geqslant 0} \big\{X\big(f(n), x\big) \dot{-} g(n)\big\}$$

that $X(f(n), x) \leqslant g(n)$, for all $n \geqslant 0$. Let $\varepsilon > 0$. Because $\inf g(n) = 0$ there is a natural number N such that $g(N) \leqslant \varepsilon$. For all $n \geqslant N$,

$$\begin{aligned} &X(f(n), x) \\ &\quad \leqslant g(n) \\ &\quad \leqslant g(N) \quad [g \text{ is decreasing}] \\ &\quad \leqslant \varepsilon. \end{aligned}$$

Thus $x = \lim f(n)$.

($\Rightarrow$) Let $\varepsilon > 0$ and let $y \in X$. For $n \geqslant 0$ and $k \geqslant 0$,

$$\begin{aligned} &X\big(f(n), y\big) \\ &\quad \leqslant X\big(f(n), f(n+k)\big) + X\big(f(n+k), x\big) + X(x,y) \quad [\text{triangle inequality}] \\ &\quad \leqslant g(n) + \varepsilon + X(x,y), \end{aligned}$$

for k big enough. Since ε was arbitrary, this implies

$$X(f(n), y) \leqslant g(n) + X(x, y),$$

for all $n \geqslant 0$, which is equivalent to

$$\sup_{n \geqslant 0}\{X(f(n), y) \dot{-} g(n)\} \leqslant X(x, y).$$

Conversely,

$$\begin{aligned}
&X(x, y)\\
&\quad \leqslant X(x, f(n)) + X(f(n), y)\\
&\quad = X(f(n), x) + X(f(n), y) \quad \text{[symmetry]}\\
&\quad \leqslant X(f(n), x) + (X(f(n), y) \dot{-} g(n)) + g(n)\\
&\quad \leqslant \varepsilon/2 + (X(f(n), y) \dot{-} g(n)) + \varepsilon/2 \quad \text{[for } n \text{ big enough]}\\
&\quad \leqslant \varepsilon + \sup_{n \geqslant 0}\{X(f(n), y) \dot{-} g(n)\}.
\end{aligned}$$

Since ε was arbitrary, this implies

$$X(x, y) \leqslant \sup_{n \geqslant 0}\{X(f(n), y) \dot{-} g(n)\}. \qquad \square$$

Although this equivalence holds only for symmetric metric spaces, the alternative definition of limit (3′) makes perfect sense for nonsymmetric, so-called generalized metric spaces X as well. For instance, any partially ordered set can be represented as a generalized metric space (via $P(p, q) = 0$, if $p \leqslant q$, and $= 1$, otherwise); then (3′) amounts to the definition of least upper bound.

Definition (3′) is an instance of the enriched-categorical notion of *weighted colimit* (or indexed colimit) [2], and makes it possible to continue Lawvere's approach to the theory of metric spaces [8], by applying further insights from enriched category theory, in particular, various results on weighted colimits (and their dual, weighted limits) [7,3]. As a consequence, many of the recently proposed definitions of generalized metric limit turn out to be theorems in enriched category theory. Furthermore, many other types of 'limits', such as the least upper bound of a directed subset or the limit of a Cauchy net, are expressible as weighted (co)limits as well.

The connection between limits of Cauchy sequences and weighted colimits is briefly mentioned at the end of [11]. Here it is worked out in further detail. The role of weights as modulus of convergence is also mentioned in [14].

(B) An interesting example of a nontrivial generalized metric space is the space of *metrical predicates* (or 'fuzzy subsets'), which is given, for any generalized metric space X by

$$\widehat{X} = [0, \infty]^{X^{op}},$$

the set of all nonexpansive functions from X^{op} to $[0, \infty]$. (Here the (nonsymmetric) distance on $[0, \infty]$ is again $\dot{-}$, i.e., truncated subtraction; and X^{op} is as X but with distance $X^{op}(x, y) = X(y, x)$.) The importance of the space $\widehat{X}$ for the study of metric spaces—strongly emphasized by Lawvere [8]—is established first and foremost by the existence of the metric *Yoneda embedding*:

$$\widehat{\boldsymbol{y}} : X \to \widehat{X}, \ x \mapsto \widehat{x} = \lambda z \in X.\, X(z, x),$$

which is isometric by the (metric) *Yoneda lemma*. As a consequence, $\widehat{X}$ has many pleasant properties:

- comparing the elements of $\widehat{X}$ with 'real' subsets of X gives rise to elementary definitions of the Hausdorff distance and the metric ε-ball topology [8,9];
- the metric Cauchy completion of X can be defined as a subspace of $\widehat{X}$ [1];
- $\widehat{X}$ gives rise to the definition of a generalized Scott topology [1];
- generalized lower and upper powerdomains can be defined as subspaces of $\widehat{X}$ and its dual $\check{X} = ([0, \infty]^X)^{op}$ [1].

Here we want to give another illustration of the beauty of $\widehat{X}$ and $\check{X}$: Certain elements in $\widehat{X}$ correspond to closed balls [9]. As we shall see, certain elements in $\check{X}$ correspond to what have been called 'formal balls' [15]. Formal balls and closed balls will be related by the *Isbell* conjugation between $\widehat{X}$ and $\check{X}$. Formal balls, supplied with a partial order, have recently been used [4] as an approximative structure for symmetric metric spaces (cf. Lawson's [10]). Here we shall generalize some of the results of that paper. The collection of formal balls will be considered as a generalized metric space $\mathcal{F}$, inheriting the distance from $\check{X}$; the underlying ordering turns out to be the one of [15]. For symmetric spaces X, the subspace of minimal elements of $\mathcal{F}$ (with respect to this underlying ordering) is isometric to X via the co-Yoneda embedding. Furthermore, $\mathcal{F}^{op}$ is (forward-)complete if and only if X is complete, and ω-algebraic if and only if X is separable.

2. Preliminaries

A *generalized metric space* (gms for short) is a set X together with a *distance* function

$$X(-,-) : X \times X \to [0, \infty]$$

which satisfies, for all x, y, and z in X,

(a) $X(x, x) = 0$, and

(b) $X(x, z) \leqslant X(x, y) + X(y, z)$,

the so-called triangle inequality. Here $+$ is the usual addition on the extended positive real numbers (with $r + \infty = \infty + r = \infty$, for any $r \in [0, \infty]$). If X moreover satisfies

(c) if $X(x, y) = 0$ and $X(y, x) = 0$ then $x = y$,

then X is called a *quasi* metric space. If X satisfies (a), (b), and

(d) $X(x, y) = X(y, x)$,

then it is called a *symmetric* metric space. Clearly, according to these definitions, any ordinary metric space is a symmetric quasi metric space.

If two elements x and x' in a gms X have distance 0 in both directions: $X(x,x') = 0 = X(x',x)$, then x and x' are called isomorphic, denoted by $x \cong x'$. In a quasi metric space, any two isomorphic elements are equal.

Examples 2.1. Examples of generalized metric spaces are:

(1) The set A^∞ of finite and infinite words over some given set A with distance function, for v and w in A^∞,

$$A^\infty(v,w) = \begin{cases} 0 & \text{if } v \text{ is a prefix of } w, \\ 2^{-n} & \text{otherwise,} \end{cases}$$

where n is the length of the longest common prefix of v and w.

(2) Any *preorder* $\langle P, \leqslant \rangle$ (satisfying for all p, q, and r in P, $p \leqslant p$, and if $p \leqslant q$ and $q \leqslant r$ then $p \leqslant r$) can be viewed as a generalized metric space, by defining

$$P(p,q) = \begin{cases} 0 & \text{if } p \leqslant q, \\ \infty & \text{if } p \not\leqslant q. \end{cases}$$

By a slight abuse of language, any gms stemming from a preorder in this way will itself be called a preorder. If P is a *partial* order: $p \leqslant q$ and $q \leqslant p$ implies $p = q$, then the induced gms is a quasi metric space.

(3) The set $[0,\infty]$ with distance, for r and s in $[0,\infty]$,

$$[0,\infty](r,s) = s \dot{-} r,$$

where

$$s \dot{-} r = \begin{cases} s - r & \text{if } s \geqslant r, \\ 0 & \text{if } s < r. \end{cases}$$

(Note that $[0,\infty]$ is a quasi metric space.)

The distance on the space $[0,\infty]$ given above makes it, in categorical terms, a complete and cocomplete symmetric monoidal closed category. More specifically, $[0,\infty]$ is a category with as objects the nonnegative real numbers and an arrow from r to s if and only $r \geqslant s$. It is trivially complete and cocomplete (products are given by supremum and coproducts by infimum). It carries a symmetric monoidal structure given by (the *tensor* product) $+$, with 0 as neutral element. And this structure is closed in the sense that for any t in $[0,\infty]$, the function (functor) $t + -$ is left adjoint to the (*cotensor*) functor $[0,\infty](t,-)$ (defined above), because for all r and s in $[0,\infty]$:

$$t + s \geqslant r \Leftrightarrow s \geqslant r \dot{-} t.$$

Generalized metric spaces can be viewed as *categories enriched in* $[0,\infty]$, or $[0,\infty]$-categories for short. By taking this view, we follow Lawvere's [8] conception of metric spaces as $\mathcal{V}$-categories [5,7]. The main advantage of this approach is that many results from enriched category theory can be applied to metric spaces.

We just saw that any preorder induces a generalized metric space. There is also the reverse construction: any generalized metric space X induces a preordered space $\langle X, \leqslant_X \rangle$ where the so-called *underlying* ordering $\leqslant_X$ is defined by

$$x \leqslant_X y \Leftrightarrow X(x,y) = 0.$$

Note that if X is a quasi metric space then a *partial* ordering is obtained.

Applying this definition to A^∞, we obtain the usual prefix ordering. The ordering underlying $[0,\infty]$ is the reverse of the usual ordering:

$$r \leqslant_{[0,\infty]} s \Leftrightarrow r \geqslant s.$$

A mapping $f: X \to Y$ between generalized metric spaces X and Y is *nonexpansive* if for all x and x' in X,

$$Y\big(f(x), f(x')\big) \leqslant X(x, x').$$

A nonexpansive map f is *isometric* if this inequality is always an equality. Two spaces X and Y are called isometric (isomorphic) if there exists an isometric bijection between them. The *exponent* of X and Y is defined by

$$Y^X = \{f: X \to Y \mid f \text{ is nonexpansive}\},$$

with distance, for f and g in Y^X,

$$Y^X(f,g) = \sup_{x \in X}\big\{Y\big(f(x), g(x)\big)\big\}.$$

Note that the underlying ordering is pointwise:

$$f \leqslant_{Y^X} g \Leftrightarrow f(x) \leqslant_Y g(x), \quad \text{for all } x \in X.$$

The *product* $X \times Y$ consists of the Cartesian product of the sets X and Y with distance

$$(X \times Y)\big(\langle x, y\rangle,\ \langle x', y'\rangle\big) = \max\big\{X(x,x'),\ Y(y,y')\big\}.$$

The *tensor product* $X \otimes Y$ of X and Y consists again of the same carrier, but now with distance

$$(X \otimes Y)\big(\langle x, y\rangle,\ \langle x', y'\rangle\big) = X(x,x') + Y(y,y').$$

The *opposite* X^{op} of a gms X is the set X with distance

$$X^{op}(x,x') = X(x',x).$$

The distance function $X(-,-)$ is a nonexpansive mapping

$$X(-,-): X^{op} \otimes X \to [0,\infty].$$

The following properties will be often used. For all r, s, and t in $[0,\infty]$,

$$[0,\infty](r+s,\ t) = [0,\infty]\big(r,\ [0,\infty](s,t)\big),$$

$$[0,\infty](t,\ s \dot{-} r) = [0,\infty]\big(r,\ [0,\infty](t,s)\big).$$

For a gms X, $f \in [0,\infty]^X$, and $r \in [0,\infty]$, let

$$r + f = \lambda x \in X.\, r + f(x),$$

$$f \dot{-} r = \lambda x \in X.\, f(x) \dot{-} r.$$

The following equalities hold:

$$[0,\infty]^X(r+f,\ g) = [0,\infty]\big(r,\ [0,\infty]^X(f,g)\big),$$

$$[0,\infty]^X(g,\ f \dot{-} r) = [0,\infty]\big(r,\ [0,\infty]^X(g,f)\big).$$

A proof may use the following elementary facts: For any subset $S \subseteq [0,\infty]$,

$$[0,\infty](r, \sup S) = \sup_{s\in S}[0,\infty](r,s),$$

$$[0,\infty](\inf S, r) = \sup_{s\in S}[0,\infty](s,r).$$

Let the space of so-called 'fuzzy subsets' and its dual be defined by

$$\widehat{X} = [0,\infty]^{X^{op}}, \quad \check{X} = \big([0,\infty]^X\big)^{op}.$$

Note that both $\widehat{X}$ and $\check{X}$ are quasi metric spaces, because $[0,\infty]$ is. The following functions are of great importance for the theory of generalized metric spaces: The *Yoneda embedding*:

$$\widehat{\boldsymbol{y}}: X \to \widehat{X}, \quad x \mapsto \widehat{\boldsymbol{y}}(x) = \lambda z \in X.\, X(z,x),$$

and the *co-Yoneda embedding*:

$$\check{\boldsymbol{y}}: X \to \check{X}, \quad x \mapsto \check{\boldsymbol{y}}(x) = \lambda z \in X.\, X(x,z).$$

We shall often use the following shorthand:

$$\widehat{x} = \widehat{\boldsymbol{y}}(x) \quad \text{and} \quad \check{x} = \check{\boldsymbol{y}}(x).$$

Both the Yoneda embedding and its dual are isometric as a consequence of the following lemma.

Lemma 2.2. *Let X be a generalized metric space.*
(1) *The Yoneda lemma: for all $x \in X$ and $\phi \in \widehat{X}$, $\widehat{X}(\widehat{x},\ \phi) = \phi(x)$.*
(2) *The co-Yoneda lemma: for all $x \in X$ and $\psi \in \check{X}$, $\check{X}(\psi,\ \check{x}) = \psi(x)$.*

A pair of nonexpansive functions $f: X \to Y$ and $g: Y \to X$ between generalized metric spaces is *adjoint* (and f is *left adjoint* to g, denoted by $f \dashv g$) if for all $x \in X$ and $y \in Y$,

$$f(x) \leqslant_Y y \Leftrightarrow x \leqslant_X g(y).$$

(Equivalently [11], for all $x \in X$ and $y \in Y$, $Y(f(x),\ y) = X(x,\ g(y))$.)

3. Weighted limits and weighted colimits

The enriched categorical definitions of weighted limit and colimit [2] are given for the special case of $[0,\infty]$-categories, that is, generalized metric spaces. Most definitions and facts of the present section are instances of general enriched-categorical versions of them, see [7] or [3]. For all facts, elementary proofs can be given as well, some of which have been included here. In the next section, we shall show that limits of Cauchy sequences are weighted (co)limits.

Let D and X be generalized metric spaces, and let

$$f : D \to X, \quad g : D \to [0,\infty]$$

be nonexpansive functions. An element x in X is a *limit of f weighted by g*:

$$x = \lim_{\leftarrow g} f \Leftrightarrow \forall y \in X, \quad X(y,x) = [0,\infty]^D\big(g,\, X(y,f)\big),$$

where $X(y,f) : D \to [0,\infty]$ maps d in D to $X(y, f(d))$. Dually, let

$$f : D \to X, \quad g : D^{op} \to [0,\infty]$$

be nonexpansive functions. An element x in X is a *colimit of f weighted by g*:

$$x = \lim_{\to g} f \Leftrightarrow \forall y \in X, \quad X(x,y) = [0,\infty]^{D^{op}}\big(g,\, X(f,y)\big),$$

where $X(f,y) : D^{op} \to [0,\infty]$ maps d in D to $X(f(d), y)$. If for a space X all weighted limits exist (for arbitrary D, f, and g) then X is called $[0,\infty]$-*complete*. And if all weighted colimits exist then X is called $[0,\infty]$-*cocomplete*.

The notion of weighted limit is dual to that of weighted colimit in that weighted limits in X correspond to weighted colimits in X^{op} (and vice versa), which can be seen as follows: For nonexpansive $f : D \to X$ and $g : D \to [0,\infty]$, and x in X,

$$\begin{aligned} x = \lim_{\leftarrow g} f &\Leftrightarrow \forall y \in X, \quad X(y,x) = [0,\infty]^D\big(g,\, X(y,f)\big) \\ &\Leftrightarrow \forall y \in X^{op}, \quad X^{op}(x,y) = [0,\infty]^D\big(g,\, X^{op}(f,y)\big). \end{aligned} \tag{1}$$

Now observe that $f : D \to X$ is also a nonexpansive function $f : D^{op} \to X^{op}$, and that $g : D \to [0,\infty]$ is a nonexpansive function $g : (D^{op})^{op} \to [0,\infty]$, since $(D^{op})^{op} = D$. Thus formula (1) expresses that x is the *colimit* of $f : D^{op} \to X^{op}$ weighted by $g : (D^{op})^{op} \to [0,\infty]$.

If x and x' are both limits of f weighted by g, then $X(x,x') = 0$ and $X(x',x) = 0$, i.e., $x \cong x'$. Thus weighted limits, and similarly weighted colimits are unique up to isomorphism and, hence, unique in quasi metric spaces. For that reason it will be often convenient to consider quasi rather than generalized metric spaces.

Weighted limits and colimits in $[0,\infty]$ can be easily described.

Theorem 3.1. *Let D be a gms.*

(1) *For nonexpansive $f : D \to [0,\infty]$ and $g : D \to [0,\infty]$:*

$$\lim_{\leftarrow g} f = [0,\infty]^D(g,f).$$

(2) *For nonexpansive $f : D \to [0,\infty]$ and $g : D^{op} \to [0,\infty]$:*

$$\lim_{\to g} f = \inf_{d \in D}\big\{f(d) + g(d)\big\}.$$

Proof. For all $y \in [0,\infty]$,

$$\begin{aligned} &[0,\infty]^D\big(g,\, [0,\infty](y,f)\big) \\ &\quad = \sup_{d \in D} [0,\infty]\big(g(d),\, [0,\infty]\big(y, f(d)\big)\big) \end{aligned}$$

$$\begin{aligned}
&= \sup_{d\in D}[0,\infty]\big(y,\ [0,\infty]\big(g(d), f(d)\big)\big)\\
&= [0,\infty]\Big(y,\ \sup_{d\in D}[0,\infty]\big(g(d), f(d)\big)\Big)\\
&= [0,\infty]\big(y,\ [0,\infty]^D(g,f)\big),
\end{aligned}$$

which proves (1). Similarly, (2) follows from

$$\begin{aligned}
&[0,\infty]^{D^{op}}\big(g,\ [0,\infty](f,y)\big)\\
&\quad= \sup_{d\in D}[0,\infty]\big(g(d),\ [0,\infty]\big(f(d),y\big)\big)\\
&\quad= \sup_{d\in D}[0,\infty]\big(f(d)+g(d),\ y\big)\\
&\quad= [0,\infty]\Big(\inf_{d\in D}\big\{f(d)+g(d)\big\},\ y\Big). \qquad \square
\end{aligned}$$

As a consequence, all weighted limits and colimits in $[0,\infty]$ exist:

Corollary 3.2. *The space $[0,\infty]$ is $[0,\infty]$-complete and $[0,\infty]$-co-complete.*

A further consequence is the following theorem, which is a special instance of [3, Proposition 6.6.17].

Theorem 3.3. *For a generalized metric space X, the space $[0,\infty]^X$ is $[0,\infty]$-complete and $[0,\infty]$-co-complete.*

A function $h: X \to Y$ between generalized metric spaces is $[0,\infty]$-*continuous* if it preserves weighted limits: i.e., for every gms D, $f: D \to X$ and $g: D \to [0,\infty]$,

$$h(\underset{\longleftarrow g}{\lim} f) \cong \underset{\longleftarrow g}{\lim}\, h \circ f.$$

Dually, $h: X \to Y$ is $[0,\infty]$-*co-continuous* if it preserves weighted colimits.

For instance, for any $y \in X$, the (nonexpansive) mappings $X(y,-): X \to [0,\infty]$ and $X(-,y): X^{op} \to [0,\infty]$ are weighted continuous [3, Proposition 6.6.11]. (Thus the latter transforms weighted colimits in X into weighted limits in $[0,\infty]$.) This is an immediate consequence of the following.

Theorem 3.4. *Let X be a generalized metric space and $x \in X$.*

(1) *For every gms D, $f: D \to X$ and $g: D \to [0,\infty]$,*

$$x = \underset{\longleftarrow g}{\lim} f \Leftrightarrow \forall y \in X,\quad X(y,x) = \underset{\longleftarrow g}{\lim} X(y,f).$$

(2) *Dually, for every gms D, $f: D \to X$ and $g: D^{op} \to [0,\infty]$,*

$$x = \underset{\longrightarrow g}{\lim} f \Leftrightarrow \forall y \in X,\quad X(x,y) = \underset{\longleftarrow g}{\lim} X(f,y).$$

Proof. We prove (1), the proof of (2) is similar. Because

$$
\begin{aligned}
x = \lim_{\leftarrow g} f &\Leftrightarrow \forall y \in X, \quad X(y,x) = [0,\infty]^D\big(g,\, X(y,f)\big) \\
&\Leftrightarrow \forall y \in X\ \forall r \in [0,\infty], \\
&\qquad [0,\infty]\big(r,\, X(y,x)\big) = [0,\infty]\big(r,\, [0,\infty]^D\big(g,\, X(y,f)\big)\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&\forall y \in X,\ X(y,x) = \lim_{\leftarrow g} X(y,f) \\
&\quad \Leftrightarrow\ \forall y \in X\ \forall r \in [0,\infty], \\
&\qquad [0,\infty]\big(r,\, X(y,x)\big) = [0,\infty]^D\big(g,\, [0,\infty]\big(r,\, X(y,f)\big)\big),
\end{aligned}
$$

the equivalence follows from

$$
\begin{aligned}
&[0,\infty]\big(r,\, [0,\infty]^D\big(g,\, X(y,f)\big)\big) \\
&\quad = [0,\infty]^D\big(r+g,\, X(y,f)\big) \\
&\quad = [0,\infty]^D\big(g,\, [0,\infty]\big(r,\, X(y,f)\big)\big). \qquad \square
\end{aligned}
$$

4. Limits of Cauchy sequences

We look at the special case of limits and colimits of Cauchy sequences weighted by Cauchy witness functions. As it turns out, we recover the definitions of backward-limit and forward-limit in generalized metric spaces, as introduced in [12], and studied in [13,11,1].

Let $\mathbb{N}$ denote the set of natural numbers:

$$\mathbb{N} = \{0, 1, 2, \ldots\},$$

with the discrete metric. Let X be a generalized metric space. A sequence $f : \mathbb{N} \to X$ is *forward-Cauchy* in X if there exists a *forward-Cauchy witness* for f; that is, a function $g : \mathbb{N} \to [0,\infty]$ satisfying,

(1) $\forall n \geqslant 0,\ g(n) \geqslant g(n+1)$;
(2) $\inf g(n) = 0$;
(3) $\forall n \geqslant 0\ \forall k \geqslant 0,\ X(f(n), f(n+k)) \leqslant g(n)$.

Dually, a sequence f is *backward-Cauchy* in X if it is forward-Cauchy in X^{op}, that is, if there exists a *backward-Cauchy* witness $g : \mathbb{N} \to [0,\infty]$, satisfying,

(1) $\forall n \geqslant 0,\ g(n) \geqslant g(n+1)$;
(2) $\inf g(n) = 0$;
(3) $\forall n \geqslant 0\ \forall k \geqslant 0,\ X(f(n+k), f(n)) \leqslant g(n)$.

A sequence is *bi-Cauchy* if it is both forward-Cauchy and backward-Cauchy.

These definitions are equivalent to the more traditional formulations:

Proposition 4.1. *Let $f : \mathbb{N} \to X$ be a sequence in a gms X.*

(1) *The sequence f is forward-Cauchy if and only if*

$$\forall \varepsilon > 0\ \exists N \geqslant 0\ \forall n \geqslant m \geqslant N, \quad X\big(f(m), f(n)\big) \leqslant \varepsilon.$$

(2) *The sequence f is backward-Cauchy if and only if*

$$\forall \varepsilon > 0\ \exists N \geqslant 0\ \forall n \geqslant m \geqslant N, \quad X\big(f(n), f(m)\big) \leqslant \varepsilon.$$

For symmetric metric spaces, forward- and backward-Cauchy means Cauchy in the usual sense. For partial orders, a forward-Cauchy sequence is an eventually increasing chain, and a backward-Cauchy sequence is an eventually decreasing chain.

Weighted colimits of forward-Cauchy sequences and weighted limits of backward-Cauchy sequences are of particular importance. We introduce the following terminology: A colimit x of a forward-Cauchy sequence $f : \mathbb{N} \to X$ weighted by a forward-Cauchy witness $g : \mathbb{N} \to X$ for f:

$$x = \lim_{\rightarrow g} f,$$

is called a *forward-limit* of f. (Note that $\mathbb{N} = \mathbb{N}^{op}$.) In that case, we shall also say that f is *forward-convergent* to x. Dually, a limit x of a backward-Cauchy sequence $f : \mathbb{N} \to X$ weighted by a backward-Cauchy witness $g : \mathbb{N} \to X$ for f:

$$x = \lim_{\leftarrow g} f,$$

is called a *backward-limit* of f.

The definitions of forward- and backward-limits do not depend on the specific choice of the witness g. This is an immediate consequence of the following lemma.

Lemma 4.2. *Consider $f : \mathbb{N} \to [0, \infty]$.*

(1) *If f is forward-Cauchy in $[0, \infty]$ and $g : \mathbb{N} \to [0, \infty]$ is a forward-Cauchy witness for f then*

$$\lim_{\rightarrow g} f = \limsup_{n \geqslant 0} f(n) \quad \Big(= \inf_{n \geqslant 0} \sup_{i \geqslant n} f(i)\Big).$$

(2) *If f is backward-Cauchy in $[0, \infty]$ and $g : \mathbb{N} \to [0, \infty]$ is a backward-Cauchy witness for f then*

$$\lim_{\leftarrow g} f = \liminf_{n \geqslant 0} f(n) \quad \Big(= \sup_{n \geqslant 0} \inf_{i \geqslant n} f(i)\Big).$$

Proof. We prove only (1), the proof of (2) is similar. By Theorem 3.1(2), it is sufficient to show

$$\inf_{n \geqslant 0} \big\{ f(n) + g(n) \big\} = \inf_{n \geqslant 0} \sup_{i \geqslant n} f(i).$$

Because

$$f(n + k) \dot{-} f(n) \leqslant g(n), \quad \text{for all } n \geqslant 0,\ k \geqslant 0,$$

which is equivalent to $f(n + k) \leqslant f(n) + g(n)$, it follows that

$$\sup_{i \geqslant n} f(i) \leqslant f(n) + g(n),$$

which implies

$$\inf_{n\geqslant 0}\sup_{i\geqslant n} f(i) \leqslant \inf_{n\geqslant 0}\{f(n)+g(n)\}.$$

Conversely, because

$$f(n)+g(n) \leqslant \Big(\sup_{i\geqslant n} f(i)\Big) + g(n),$$

we have

$$\begin{aligned}
&\inf_{n\geqslant 0}\{f(n)+g(n)\}\\
&\quad\leqslant \inf_{n\geqslant 0}\Big\{\Big(\sup_{i\geqslant n} f(i)\Big)+g(n)\Big\}\\
&\quad\leqslant \inf_{n\geqslant 0}\sup_{i\geqslant n} f(i) + \inf_{n\geqslant 0} g(n) && [\text{because both } (\sup_{i\geqslant n} f(i))_n \text{ and}\\
&&& (g(n))_n \text{ are decreasing}]\\
&\quad= \inf_{n\geqslant 0}\sup_{i\geqslant n} f(i) && [\inf_{n\geqslant 0} g(n)=0]. \qquad \square
\end{aligned}$$

Corollary 4.3. *Let $f:\mathbb{N}\to X$ be a sequence in a generalized metric space X and $x\in X$.*

(1) *If $g:\mathbb{N}\to[0,\infty]$ and $g':\mathbb{N}\to[0,\infty]$ are forward-Cauchy witnesses for f, then*

$$x = \lim_{\to g} f \Leftrightarrow x = \lim_{\to g'} f.$$

(2) *Similarly for backward-Cauchy sequences and backward-limits.*

Proof. We only prove the first part, the second being dual:

$$\begin{aligned}
x = \lim_{\to g} f &\Leftrightarrow \forall y\in X,\ X(x,y) = \lim_{\leftarrow g} X(f,y) && [\text{Theorem 3.4}]\\
&\Leftrightarrow \forall y\in X,\ X(x,y) = \liminf_{n\geqslant 0} X(f(n),y) && [\text{Lemma 4.2}]\\
&\Leftrightarrow \forall y\in X,\ X(x,y) = \lim_{\leftarrow g'} X(f,y) && [\text{Lemma 4.2}]\\
&\Leftrightarrow x = \lim_{\to g'} f && [\text{Theorem 3.4}]. \qquad \square
\end{aligned}$$

The following notation is now justified: for a forward-Cauchy sequence $f:\mathbb{N}\to X$ and $x\in X$,

$$x = \lim_{\to} f \Leftrightarrow x \text{ is a forward-limit of } f \Leftrightarrow x = \lim_{\to g} f,$$
$$\text{for every forward-Cauchy witness } g \text{ for } f \quad [\text{Corollary 4.3}].$$

And, dually, for a backward-Cauchy sequence $f:\mathbb{N}\to X$ and $x\in X$,

$$x = \lim_{\leftarrow} f \Leftrightarrow x \text{ is a backward-limit of } f \Leftrightarrow x = \lim_{\leftarrow g} f,$$
$$\text{for every backward-Cauchy witness } g \text{ for } f \quad [\text{Corollary 4.3}].$$

It follows from the above (and Theorem 3.4) that forward- and backward-limits can be characterized in the following 'weightless' way.

Theorem 4.4. *For a forward-Cauchy sequence* $f:\mathbb{N}\to X$ *and* $x\in X$,

$$x=\varinjlim f \Leftrightarrow \forall y\in X,\quad X(x,y)=\varprojlim X(f,y).$$

Dually, for a backward-Cauchy sequence $f:\mathbb{N}\to X$ *and* $x\in X$,

$$x=\varprojlim f \Leftrightarrow \forall y\in X,\quad X(y,x)=\varprojlim X(y,f).$$

The above characterization of forward-limit and backward-limit has been taken as a *definition* in some recent papers on the reconciliation of the domain theories of partial orders and metric spaces [12,13,11,1]. For symmetric metric spaces, the notions of forward-limit and backward-limit are equivalent, and coincide with the standard definition of metric limit (Theorem 1.1). We have seen that for partial orders, a forward-Cauchy sequence is an eventually increasing chain; a forward-limit is a least upper bound of the chain. Backward-limits correspond to greatest lower bounds of (eventually) decreasing chains. (Cf. [1].)

A space X is *forward-complete* if every forward-Cauchy sequence in X has a forward-limit in X. Dually, X is backward-complete whenever X^{op} is forward-complete. For instance, $[0,\infty]$ is both $[0,\infty]$-complete and $[0,\infty]$-co-complete (Corollary 3.2). In particular, it is forward-complete and backward-complete. The same applies, for any gms X, to the space $\widehat{X}$ of metric predicates (Theorem 3.3). Also the space A^∞ is both forward- and backward-complete.

For any gms X, the *forward-completion* [1, Definition 5.1] $\overline{X}$ of a gms X is defined by

$$\overline{X}=\bigcap\{V\subseteq\widehat{X}\mid \widehat{\boldsymbol{y}}(X)\subseteq V \text{ and } V \text{ is a forward-complete subspace of } \widehat{X}\}.$$

(Recall that $\widehat{\boldsymbol{y}}:X\to\widehat{X}$ is the Yoneda embedding.) Because $\widehat{X}$ is a forward-complete quasi metric space, so is $\overline{X}$. It has the usual universal property [1, Theorem 5.5].

A mapping $h:X\to Y$ between generalized metric spaces is *forward-continuous* if it preserves forward-Cauchy sequences and their forward-limits: that is, if $f:\mathbb{N}\to X$ is forward-Cauchy then $h\circ f:\mathbb{N}\to Y$ is again forward-Cauchy, and any forward-limit of f is mapped by h to a forward-limit of $h\circ f$. Schematically:

$$h(\varinjlim f)\cong\varinjlim h\circ f.$$

Backward-continuity of $h:X\to Y$ is defined dually, denoted by

$$h(\varprojlim f)\cong\varprojlim h\circ f.$$

For symmetric metric spaces, forward-continuity and backward-continuity are equivalent to the usual notion of metric continuity. For partial orders, forward-continuity means preservation of least upper bounds of ascending chains, and backward-continuity is its dual.

It may be worthwhile to contrast the definition of forward-continuity with the following condition, based on the notion of $[0,\infty]$-co-continuity (Section 3): for every forward-

Cauchy sequence $f:\mathbb{N} \to X$ and every forward-Cauchy witness $g:\mathbb{N} \to [0,\infty]$ for f,

$$h(\lim_{\to g} f) \cong \lim_{\to g} h \circ f.$$

(This could be called: $[0,\infty]$-co-continuity with respect to forward-Cauchy sequences.) This condition is stronger than forward-continuity, since it requires h to preserve forward-limits with respect to the same Cauchy witness g (notably, g should again be a witness for $h \circ f$). This property is satisfied, for instance, by nonexpansive functions between symmetric metric spaces, but not by the function $h:[0,\infty] \to [0,\infty]$ defined by $h(r) = 2 \times r$, which *is* forward-continuous.

The following lemma [1, Proposition 3.1] will be useful in the sequel.

Lemma 4.5. *The distance function*

$$[0,\infty][-,-]:[0,\infty]^{op} \otimes [0,\infty] \to [0,\infty]$$

is both forward- and backward-continuous.

Also the following fact will be used, which states that in $[0,\infty]$, backward- and forward-Cauchy sequences are closely related.

Lemma 4.6. *Any forward-Cauchy sequence in $[0,\infty]$ is also backward-Cauchy. The reverse holds for those sequences $f:\mathbb{N} \to [0,\infty]$ which are* bounded: *there exists K in $\mathbb{N}$ with $f(n) \leqslant K$, for all $n \geqslant 0$.*

The boundedness condition in the lemma above is to exclude sequences like $(0,1,2,\ldots)$, which is backward-Cauchy but not forward-Cauchy.

Even though the metric on $[0,\infty]$ is not symmetric, limits in $[0,\infty]$ are as we are used to:

Corollary 4.7. *Every sequence $f:\mathbb{N} \to [0,\infty]$ that is both forward-Cauchy and backward-Cauchy has a forward-limit and a backward-limit, which coincide and are equal to the limit of the sequence f with respect to the standard Euclidean distance*:

$$|r - s| = \max\{(r \dot{-} s),\ (s \dot{-} r)\}.$$

Lemma 4.6 and Corollary 4.7 can be easily proved. They also follow from [12, Theorem 10].

5. Algebraicity

We briefly recall from [1] the definition of algebraic generalized metric space together with the observation that $[0,\infty]$ is algebraic. This will be used in Section 7.

An element k in a gms X is *finite* (or *compact*) in X if the mapping

$$X(k,-):X \to [0,\infty], \quad x \mapsto X(k,x)$$

is forward-continuous: that is, for all forward-convergent sequences $f:\mathbb{N}\to X$,

$$X(k,\underrightarrow{\lim} f) = \underrightarrow{\lim}\, X(k,f).$$

If X is a partial order this means that for any chain $(x_n)_n$ in X,

$$X\Bigl(k,\bigsqcup x_n\Bigr) = \underleftarrow{\lim}\, X(k,x_n),$$

or, equivalently,

$$k \leqslant_X \bigsqcup x_n \quad \text{iff} \quad \exists n,\ k \leqslant_X x_n,$$

which is the usual definition of finiteness in ordered spaces. If X is a symmetric metric space then $X(k,-)$ is forward-continuous for any k in X, hence all elements are finite.

A *basis* for a gms X is a subset $K \subseteq X$ consisting of finite elements such that every element x in X is the forward-limit of a forward-Cauchy sequence $(k_n)_n$ of elements in K. A gms X is *algebraic* if there exists a basis for X. Note that such a basis is in general not unique. If X is algebraic then the collection K_X of all finite elements of X is the largest basis. If there exists a countable basis then X is *ω-algebraic*.

Any symmetric metric space is algebraic, because all elements are finite. If the space is separable then it is ω-algebraic. The gms A^∞ from Section 2 is algebraic with basis A^*, the set of all finite words over A. If A is countable then A^∞ is ω-algebraic.

Also the space $[0,\infty]$ is algebraic: by Lemma 4.5, all elements are finite. (It is even ω-algebraic, with the set of nonnegative rational numbers as a basis.) This fact is somewhat surprising, since $[0,\infty]$ is not algebraic as a partial order.

Every forward-complete quasi metric space X with basis K, is isomorphic to the completion of its basis: $X \cong \overline{K}$. For a proof, see [1, Theorem 5.6].

6. Balls: formal, fuzzy, closed

Let X be a generalized metric space. We repeat from [9] the definition of the subspace of $\widehat{X}$ consisting of (fuzzy) balls. Dually, we introduce as a subspace of $\check{X}$ a collection of so-called formal balls. The partial order underlying this subspace of formal balls is shown to be isomorphic with the partially ordered set of formal balls introduced in [15]. Moreover, balls and formal balls will be related by means of the Isbell conjugation between $\widehat{X}$ and $\check{X}$. In Section 7, the collection of formal balls will be shown to be a computational model for ordinary metric spaces.

Let for any $r \in [0,\infty]$ and $x \in X$,

$$B\langle r,x\rangle = \widehat{x} \dot{-} r \quad (= \lambda y \in X.\, X(y,x) \dot{-} r),$$

$$F\langle r,x\rangle = r + \check{x} \quad (= \lambda y \in X.\, r + X(x,y)),$$

and define two parameterized families $\mathcal{B} \subseteq \widehat{X}$ and $\mathcal{F} \subseteq \check{X}$ by

$$\mathcal{B} = \bigl\{B\langle r,x\rangle \mid r \in [0,\infty],\ x \in X\bigr\},$$

$$\mathcal{F} = \bigl\{F\langle r,x\rangle \mid r \in [0,\infty],\ x \in X\bigr\}.$$

The sets $\mathcal{B}$ and $\mathcal{F}$ are taken with the distance inherited from $\widehat{X}$ and $\check{X}$, respectively. The elements $B\langle r, x\rangle$ of $\mathcal{B}$ are called *fuzzy balls*, and the elements $F\langle r, x\rangle$ of $\mathcal{F}$ are called *formal balls*, for reasons to be explained next.

To start with the former, we recall [9,1] the following adjunction between the collection $\mathcal{P}(X)$ of subsets of X, and $\widehat{X}$:

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "\rho"] \arrow[r, phantom, "\perp"] & \widehat{X}, \arrow[l, bend left, "\int"] \end{tikzcd}$$

defined for $\phi \in \widehat{X}$ and $V \in \mathcal{P}(X)$ by

$$\int \phi = \{x \in X \mid \phi(x) = 0\}, \quad \rho(V) = \lambda x \in X.\ \inf_{v \in V} X(x, v).$$

Applying $\int$ to $B\langle r, x\rangle$ yields

$$\begin{aligned} \int B\langle r, x\rangle &= \{y \in X \mid B\langle r, x\rangle(y) = 0\} \\ &= \{y \in X \mid X(y, x) \dot{-} r = 0\} \\ &= \{y \in X \mid X(y, x) \leqslant r\} \\ &= B_r(x), \end{aligned}$$

the *closed ball* with centre x and radius r [9, p. 171].

The connection between elements $F\langle r, x\rangle \in \mathcal{F}$ and formal balls in the sense of [15] can be understood by looking at the ordering underlying $\mathcal{F}$. First we note that for $F\langle r, x\rangle$ and $F\langle s, y\rangle$ in $\mathcal{F}$,

$$\begin{aligned} &\mathcal{F}(F\langle r, x\rangle,\ F\langle s, y\rangle) \\ &\quad = \check{X}(r + \check{x},\ s + \check{y}) \\ &\quad = \left([0, \infty]^X\right)^{op}(r + \check{x},\ s + \check{y}) \\ &\quad = [0, \infty]^X(s + \check{y},\ r + \check{x}) \\ &\quad = [0, \infty]^X(\check{y},\ (r + \check{x}) \dot{-} s) \\ &\quad = \check{X}((r + \check{x}) \dot{-} s,\ \check{y}) \\ &\quad = ((r + \check{x}) \dot{-} s)(y) \quad \text{[co-Yoneda Lemma 2.2]} \\ &\quad = (r + X(x, y)) \dot{-} s. \end{aligned}$$

Lemma 6.1. *For $F\langle r, x\rangle$ and $F\langle s, y\rangle$ in $\mathcal{F}$,*

$$\mathcal{F}(F\langle r, x\rangle,\ F\langle s, y\rangle) = (r + X(x, y)) \dot{-} s.$$

Consequently, the ordering $\leqslant_{\mathcal{F}}$ underlying $\mathcal{F}$ can be characterized by

$$\begin{aligned} &F\langle r, x\rangle \leqslant_{\mathcal{F}} F\langle s, y\rangle \\ &\quad \Leftrightarrow \mathcal{F}(F\langle r, x\rangle,\ F\langle s, y\rangle) = 0 \\ &\quad \Leftrightarrow (r + X(x, y)) \dot{-} s = 0 \\ &\quad \Leftrightarrow r + X(x, y) \leqslant s, \end{aligned}$$

which is precisely the definition of the ordering on formal balls from [15, p. 7].

Formal balls and closed balls can be related by the so-called *Isbell conjugation*, which is recalled first [9, p. 169]: for $\phi \in \widehat{X}$ and $\psi \in \check{X}$ let

$$(\phi)^* = \lambda x \in X.\, \widehat{X}(\phi, \widehat{x}),$$

$$(\psi)^\sharp = \lambda x \in X.\, \check{X}(\check{x}, \psi).$$

The functions $()^*$ and $()^\sharp$ are nonexpansive; moreover, $()^*$ is left adjoint to $()^\sharp$:

$$\begin{tikzcd} \widehat{X} \arrow[r, bend left, "()^*"] \arrow[r, phantom, "\perp"] & \check{X}. \arrow[l, bend left, "()^\sharp"] \end{tikzcd}$$

Proposition 6.2. *For $F\langle r, x\rangle$ in $\mathcal{F}$,*

$$\big(F\langle r, x\rangle\big)^\sharp = B\langle r, x\rangle.$$

Applying the function $\int$ on both sides yields $\int(F\langle r, x\rangle)^\sharp = B_r(x)$.

Proof. The second equality is immediate from the first one, which is proved as follows: for all $y \in X^{op}$,

$$\begin{aligned} &\big(F\langle r, x\rangle\big)^\sharp(y) \\ &\quad = (r + \check{x})^\sharp(y) \\ &\quad = \check{X}(\check{y},\, r + \check{x}) \\ &\quad = \mathcal{F}\big(F\langle 0, y\rangle,\, F\langle r, x\rangle\big) \\ &\quad = X(y, x) \mathbin{\dot{-}} r \quad \text{[by Lemma 6.1]} \\ &\quad = B\langle r, x\rangle(y). \qquad \square \end{aligned}$$

The assignment of a formal ball to every pair $\langle r, x\rangle$ in fact defines a nonexpansive function

$$F : \big([0, \infty]^{op} \otimes X\big) \to \mathcal{F}, \quad \langle r, x\rangle \mapsto F\langle r, x\rangle$$

(recall from Section 2 that $\otimes$ is the tensor product):

$$\begin{aligned} &\mathcal{F}\big(F\langle r, x\rangle,\, F\langle s, y\rangle\big) \\ &\quad = \big(r + X(x, y)\big) \mathbin{\dot{-}} s \quad \text{[Lemma 6.1]} \\ &\quad \leqslant (r \mathbin{\dot{-}} s) + X(x, y) \\ &\quad = [0, \infty]^{op}(r, s) + X(x, y) \\ &\quad = \big([0, \infty]^{op} \otimes X\big)\big(\langle r, x\rangle,\, \langle s, y\rangle\big). \end{aligned}$$

Restricted to $[0, \infty)^{op} \otimes X$, F is a bijection. Similarly, the function

$$B : \big([0, \infty]^{op} \otimes X\big) \to \mathcal{B}, \quad \langle r, x\rangle \mapsto B\langle r, x\rangle,$$

is nonexpansive. The space X can be isometrically embedded into $\mathcal{F}$ and $\mathcal{B}$ by composing F and B, respectively, with

$$\iota : X \to [0,\infty]^{op} \otimes X, \quad x \mapsto \langle 0, x\rangle.$$

Theorem 6.3. *Both $F \circ \iota = \check{\boldsymbol{y}} : X \to \mathcal{F}$ and $B \circ \iota = \hat{\boldsymbol{y}} : X \to \mathcal{B}$ are isometric.*

All in all, we have the following diagram, in which every triangle commutes:

$$\begin{tikzcd}
\mathcal{B} & & \mathcal{F} \arrow[ll, "()^\sharp"'] \\
& {[0,\infty]^{op} \otimes X} \arrow[ul, "B"'] \arrow[ur, "F"] & \\
& X \arrow[u, "\iota"'] \arrow[uul, bend left, "\hat{\boldsymbol{y}}"] \arrow[uur, bend right, "\check{\boldsymbol{y}}"'] &
\end{tikzcd}$$

7. A computational model for metric spaces

As a variation on recent definitions and results on computational models for ordinary metric spaces [10,4,6], we show that for an ordinary metric space X, the opposite $\mathcal{F}^{op}$ of its collection of formal balls can be considered as a computational model in the following sense:

(1) X is isometric to the collection of maximal elements (with respect to the underlying ordering) of $\mathcal{F}^{op}$ by means of the co-Yoneda embedding;
(2) X is complete if and only if $\mathcal{F}^{op}$ is forward-complete;
(3) X is separable if and only if $\mathcal{F}^{op}$ is ω-algebraic.

The first claim is a consequence of the following.

Theorem 7.1. *For an ordinary metric space X, the collection of minimal elements of $\mathcal{F}$,*

$$\min \mathcal{F} = \{\phi \in \mathcal{F} \mid \forall \psi \in \mathcal{F},\ \psi \leqslant_{\mathcal{F}} \phi \Rightarrow \psi = \phi\}$$

is isometric with X: $\min \mathcal{F} \cong X$.

Proof. Since $\check{\boldsymbol{y}} = F \circ \iota$ is isometric, it is sufficient to show that $\check{\boldsymbol{y}}$ is a bijection between X and $\min \mathcal{F}$. For all x in X, $\check{\boldsymbol{y}}(x) = \check{x}$ is minimal: if $r + \check{z} \leqslant_{\mathcal{F}} \check{x}$, for some z in X and r in $[0,\infty]$, then $r + X(z,x) \leqslant 0$, whence $r = 0$ and $X(z,x) = 0$. Because X is an ordinary metric space, by assumption, it follows that $x = z$. Thus $r + \check{z} = \check{x}$. Conversely, suppose $r + \check{z}$ is minimal in $\mathcal{F}$. Since $\check{z} \leqslant_{\mathcal{F}} r + \check{z}$ it follows that $\check{z} = r + \check{z}$, thus $r + \check{z} \in \check{\boldsymbol{y}}(X)$. Clearly, the above defines a bijection. □

It follows that the image of X (in $\mathcal{F}$) with the generalized Scott topology [1] is homeomorphic to X with the ε-ball topology.

The second claim, which relates completeness of X and $\mathcal{F}^{op}$, will follow from the next lemma.

Lemma 7.2. *Let $(x_n)_n$ be a sequence in X and $(r_n)_n$ a sequence in $[0,\infty]$.*

(1) *The sequence $(F\langle r_n, x_n\rangle)_n$ is backward-Cauchy in $\mathcal{F}$ if and only if*
 (a) *$(x_n)_n$ is backward-Cauchy in X, and*
 (b) *$(r_n)_n$ is forward-Cauchy in $[0,\infty]$.*

(2) *In the situation of* (1): *for all $x \in X$ and $r \in [0,\infty)$,*

$$\big(x = \underset{\leftarrow}{\lim} x_n \text{ and } r = \underset{\rightarrow}{\lim} r_n\big) \Rightarrow F\langle r, x\rangle = \underset{\leftarrow}{\lim} F\langle r_n, x_n\rangle.$$

If X is a symmetric metric space or if the forward-limit of $(r_n)_n$ is 0, then the converse implication holds as well.

(3) *If $(r_n)_n$ is bounded, then the dual of* (1) *and* (2) *(interchanging forward and backward) hold as well.*

Proof. We prove (1) and (2), the proof of (3) is dual.

(1) Assuming (a) and (b), it follows that $\langle r_n, x_n\rangle_n$ is backward-Cauchy in $[0,\infty]^{op} \otimes X$. Because $F : [0,\infty]^{op} \otimes X \to \mathcal{F}$ is nonexpansive, also $(F\langle r_n, x_n\rangle)_n$ is backward-Cauchy.

For the converse, assume that $(F\langle r_n, x_n\rangle)_n$ is backward-Cauchy. For all $n \geqslant m$,

$$\begin{aligned}
&\mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle r_m, x_m\rangle\big)\\
&\quad = \big(r_n + X(x_n, x_m)\big) \dot{-} r_m\\
&\quad \geqslant r_n \dot{-} r_m\\
&\quad = [0,\infty](r_m, r_n),
\end{aligned} \tag{2}$$

which shows that $(r_n)_n$ is forward-Cauchy. By Lemma 4.6, it is also backward-Cauchy. Since equality (2) above implies

$$r_n + X(x_n, x_m) \leqslant \mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle r_m, x_m\rangle\big) + r_m,$$

we have

$$\begin{aligned}
&X(x_n, x_m)\\
&\quad \leqslant \big(\mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle r_m, x_m\rangle\big) + r_m\big) \dot{-} r_n\\
&\quad \leqslant \mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle r_m, x_m\rangle\big) + (r_m \dot{-} r_n)\\
&\quad = \mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle r_m, x_m\rangle\big) + [0,\infty](r_n, r_m),
\end{aligned}$$

which shows that $(x_n)_n$ is backward-Cauchy.

(2) Assume

$$x = \underset{\leftarrow}{\lim} x_n \quad \text{and} \quad r = \underset{\rightarrow}{\lim} r_n.$$

For all $s \in [0,\infty]$ and $y \in X$,

$$
\begin{aligned}
&\underleftarrow{\lim}\, \mathcal{F}\big(F\langle s,y\rangle,\ F\langle r_n,x_n\rangle\big) \\
&\quad = \underleftarrow{\lim}\big(\big(s+X(y,x_n)\big) \dot{-} r_n\big) \\
&\quad = \underleftarrow{\lim}\big(s+X(y,x_n)\big) \dot{-} \underrightarrow{\lim}\, r_n \quad [\dot{-} \text{ is backward-continuous}] \\
&\quad = \big(s+\underleftarrow{\lim}\, X(y,x_n)\big) \dot{-} \underrightarrow{\lim}\, r_n \\
&\quad = \big(s+X(y,x)\big) \dot{-} \underrightarrow{\lim}\, r_n \quad [\text{by Theorem 4.4}] \\
&\quad = \big(s+X(y,x)\big) \dot{-} r \\
&\quad = \mathcal{F}\big(F\langle s,y\rangle,\ F\langle r,x\rangle\big).
\end{aligned}
$$

Again by Theorem 4.4, this implies

$$F\langle r,x\rangle = \underleftarrow{\lim}\, F\langle r_n,x_n\rangle.$$

For the converse, assume that the latter equality holds, for certain r in $[0,\infty]$ and x in X. It follows from the above that for all s in $[0,\infty]$ and y in X,

$$\big(s+\underleftarrow{\lim}\ X(y,x_n)\big) \dot{-} \underrightarrow{\lim}\, r_n = \big(s+X(y,x)\big) \dot{-} r.$$

Taking in this equation

$$s = r + \underrightarrow{\lim}\, r_n$$

yields

$$r + \underleftarrow{\lim}\, X(y,x_n) = \underrightarrow{\lim}\, r_n + X(y,x). \tag{3}$$

If

$$\underrightarrow{\lim}\, r_n = 0$$

then taking $y = x$ in Eq. (3) yields $r = 0$, whence

$$\underleftarrow{\lim}\, X(y,x_n) = X(y,x),$$

which by Theorem 4.4 is equivalent to

$$x = \underleftarrow{\lim}\, x_n.$$

Otherwise, assume that the space X is symmetric. Then the sequence $(x_n)_n$ is bi-Cauchy. Taking $y = x$ in Eq. (3) now gives

$$r + \underleftarrow{\lim}\, X(x,x_n) = \underrightarrow{\lim}\, r_n, \tag{4}$$

which implies

$$\underleftarrow{\lim}\, X(y,x_n) = \underleftarrow{\lim}\, X(x,x_n) + X(y,x). \tag{5}$$

Taking $y = x_k$ in Eq. (5) yields

$$\begin{aligned}&\lim_{\leftarrow} X(x, x_n)\\&\quad\leqslant \lim_{\leftarrow} X(x, x_n) + X(x_k, x)\\&\quad= \lim_{\leftarrow} X(x_k, x_n),\end{aligned}$$

where in the formulae above the limit is taken with respect to n. Because $(x_n)_n$ is bi-Cauchy, the latter number becomes arbitrarily small (for large k), whence

$$\lim_{\leftarrow} X(x, x_n) = 0.$$

It now follows from Eq. (4), and from Eq. (3) and Theorem 4.4 that

$$r = \lim_{\rightarrow} r_n \quad \text{and} \quad x = \lim_{\leftarrow} x_n. \qquad \square$$

The following theorem is an immediate consequence.

Theorem 7.3. *A generalized metric space X is backward-complete if and only if the space $\mathcal{F}$ is.*

Proof. Suppose that X is backward-complete. Let $(F\langle r_n, x_n\rangle)_n$ be a backward-Cauchy sequence in $\mathcal{F}$. By Lemma 7.2(1), $(r_n)_n$ is forward-Cauchy and $(x_n)_n$ is backward-Cauchy. By Lemma 7.2(2),

$$\lim_{\leftarrow} F\langle r_n, x_n\rangle = F\langle \lim_{\rightarrow} r_n, \lim_{\leftarrow} x_n\rangle.$$

Thus $\mathcal{F}$ is backward-complete. For the converse, consider a backward-Cauchy sequence $(x_n)_n$ in X. By Lemma 7.2(1), $(F\langle 0, x_n\rangle)_n$ is backward-Cauchy in $\mathcal{F}$. Let $F\langle r, x\rangle$ be its backward-limit. By Lemma 7.2(2), it follows that ($r = 0$ and)

$$\lim_{\leftarrow} x_n = x.$$

Thus X is backward-complete. $\square$

In particular, a symmetric metric space X is complete if and only if $\mathcal{F}^{op}$ is forward-complete.

Finally, an ordinary metric space X is separable if and only if $\mathcal{F}^{op}$ is ω-algebraic. This will be a consequence of the following lemma.

Lemma 7.4. *If X is a symmetric metric space then $\mathcal{F}^{op}$ is algebraic: all elements in $\mathcal{F}^{op}$ are finite.*

Proof. We have to show that for an element $F\langle s, y\rangle$ in $\mathcal{F}^{op}$, the function $\mathcal{F}^{op}(F\langle s, y\rangle, -)$ is forward-continuous. That is, for any forward-Cauchy sequence $(F\langle r_n, x_n\rangle)_n$ in $\mathcal{F}^{op}$,

$$\mathcal{F}^{op}\big(F\langle s, y\rangle, \lim_{\rightarrow} F\langle r_n, x_n\rangle\big) = \lim_{\rightarrow} \mathcal{F}^{op}\big(F\langle s, y\rangle, F\langle r_n, x_n\rangle\big).$$

Since forward-Cauchy and forward-limit in $\mathcal{F}^{op}$ means backward-Cauchy and backward-limit in $\mathcal{F}$, this is equivalent to

$$\mathcal{F}\big(\lim_{\leftarrow} F\langle r_n, x_n\rangle, F\langle s, y\rangle\big) = \lim_{\rightarrow} \mathcal{F}\big(F\langle r_n, x_n\rangle, F\langle s, y\rangle\big).$$

The latter equality follows from

$$\begin{aligned}
&\mathcal{F}\big(\underleftarrow{\lim}\, F\langle r_n, x_n\rangle,\ F\langle s, y\rangle\big) \\
&= \mathcal{F}\big(F\langle \underrightarrow{\lim}\, r_n,\ \underleftarrow{\lim}\, x_n\rangle,\ F\langle s, y\rangle\big) && \text{[Lemma 7.2(2)]} \\
&= \big(\underrightarrow{\lim}\, r_n + X(\underleftarrow{\lim}\, x_n,\ y)\big) \dot{-} s \\
&= \big(\underrightarrow{\lim}\, r_n + X(y, \underrightarrow{\lim}\, x_n)\big) \dot{-} s && [X \text{ is symmetric}] \\
&= \big(\underrightarrow{\lim}\, r_n + \underrightarrow{\lim}\, X(y, x_n)\big) \dot{-} s && [X \text{ is symmetric, thus } y \text{ is finite}] \\
&= \underrightarrow{\lim}\big(r_n + X(y, x_n)\big) \dot{-} s && [+ \text{ and } \dot{-} \text{ are forward-continuous}] \\
&= \underrightarrow{\lim}\, \big(r_n + X(x_n, y)\big) \dot{-} s && [X \text{ is symmetric}] \\
&= \underrightarrow{\lim}\, \mathcal{F}\big(F\langle r_n, x_n\rangle,\ F\langle s, y\rangle\big). && \square
\end{aligned}$$

Corollary 7.5. *An ordinary metric space X is* separable *if and only if $\mathcal{F}^{op}$ is ω-algebraic.*

Proof. If X is separable then it has a countable basis $V \subseteq X$. Let

$$Q = \{q \in [0, \infty] \mid q \text{ is rational}\}.$$

It follows from Lemmas 7.2 and 7.4 that $Q \times V$ is a countable basis for $\mathcal{F}^{op}$. If, conversely, B is a countable base for $\mathcal{F}^{op}$, then the set

$$\{x \in X \mid \exists r \in [0, \infty],\ \langle r, x\rangle \in B\}$$

is a countable base for X, hence X is separable. $\square$

The latter observation on separable metric spaces is to be contrasted with [4, Corollary 2.10], stating that an ordinary metric space X is separable if and only if the *poset* $\mathcal{F}^{op}$ is *ω-continuous*.

8. Conclusions and directions

(A) The use of weighted limits and colimits in gms's gives a purely enriched-categorical formulation of forward- and backward-Cauchy sequences and their limits. Thus a categorical foundation has been provided for generalized metric analysis.

Also in the standard analysis of real numbers, the formulation of metric limits as weighted colimits may have some advantages. Consider, for instance, two ordinary, i.e., bi-Cauchy sequences $f, g : \mathbb{N} \to [0, \infty]$, with (bi-)limits $x = \lim f$ and $y = \lim g$. The following is a standard observation in any basic course on analysis:

$$\text{if } \forall n \geqslant 0,\ f(n) \geqslant g(n) \quad \text{then } x \geqslant y.$$

A standard elementary proof derives from the assumption that $x < y$ the fact that $f(n) < g(n)$, for some n big enough. The following direct proof in terms of weighted colimits exploits the fact that $[0, \infty]$ has a nontrivial underlying ordering. If k and l are (forward-)Cauchy witnesses for the sequences f and g then $m = k + l$ is is common witness for both f and g. Now:

$$\begin{aligned}
&[0,\infty](x,y) \\
&\quad = [0,\infty]^{\mathbb{N}}\big(m,\ [0,\infty](f,y)\big) \quad [\text{since } x = \varinjlim_{m} f] \\
&\quad \leqslant [0,\infty]^{\mathbb{N}}\big(m,\ [0,\infty](g,y)\big) \quad [\text{since } [0,\infty](-,y) \text{ is anti-monotone}] \\
&\quad = [0,\infty](y,y) \qquad\qquad\qquad [\text{since } y = \varinjlim_{m} g] \\
&\quad = 0,
\end{aligned}$$

which implies $x \geqslant y$. In a similar way, the use of weights leads to very simple proofs of facts such as: if $x = \lim f$ and $x \neq 0 \neq f$ then $1/x = \lim 1/f$, for which the traditional argument amounts to a somewhat cumbersome calculation involving 'ε'. It would be interesting to see where the development of a categorical, 'ε-less' form of analysis will lead us.

(B) Formal balls arise in a natural way in the world of generalized metric spaces by means of the co-Yoneda embedding, and are related to fuzzy and closed balls by means of the Isbell conjugation. The (opposite of the) collection of formal balls is a computational model for ordinary metric spaces.

Our definition of computational model is different from the one in [10] and [4], though. In particular, the space $\mathcal{F}^{op}$ is equipped with a nonsymmetric distance, from which the usual ordering on formal balls (of [15]) can be retrieved as the underlying ordering. This leads to the stronger result that an ordinary metric space is *isometric* and not only *homeomorphic* with the collection of maximal elements.

(C) It may be worthwhile to investigate the use of weight functions and generalized metric spaces as computational models for ordinary metric spaces somewhat further still. In particular, if $f : \mathbb{N} \to [0,\infty]$ is a bi-Cauchy sequence with witness function g and limit $x = \lim f$, then

$$\varinjlim_{g} f = x = \varprojlim_{g} f,$$

which implies

$$x \dot{-} g \leqslant f \leqslant x + g.$$

Note that in this way we have approximated f from the left by a monotone increasing, and from the right by a monotone decreasing function. (In other words, a descending chain and a chain in $\mathcal{F}^{op}$, respectively.) These simple approximations could be of help in a theory of approximation of metric spaces in the style of [10] and [4].

Acknowledgements

Many thanks to Franck van Breugel, Reinhold Heckmann, and Kim Wagner for their comments on an earlier version of this paper. The comments by the anonymous referee are also gratefully acknowledged.

References

[1] M.M. Bonsangue, F. van Breugel and J.J.M.M. Rutten, Generalized metric spaces: completion, topology, and powerdomains via the Yoneda embedding, Report CS-R9636, CWI (1996); also: Theoretical Computer Science 193 (1998) 1–51. FTP-available via www.cwi.nl/~janr. This report extends CS-R9560, where similar results are given for ultrametric spaces.

[2] F. Borceux and G.M. Kelly, A notion of limit for enriched categories, Bull. Austral. Math. Soc. 12 (1975) 49–72.

[3] F. Borceux, Handbook of Categorical Algebra 2: Categories and Structures, Encyclopedia Math. Appl. 51 (Cambridge University Press, 1994).

[4] A. Edalat and R. Heckmann, A computational model for metric spaces, Theoretical Computer Science 193 (1998) 53–73.

[5] S. Eilenberg and G.M. Kelly, Closed categories, in: S. Eilenberg, D.K. Harrison, S. MacLane and H. Röhrl, eds., Proceedings of the Conference on Categorical Algebra (Springer, Berlin, 1966) 421–562.

[6] B. Flagg and R. Kopperman, Computational models for ultrametric spaces, in: Proceedings MFPS'97, Electronic Notes in Theoret. Comput. Sci. (1997); URL: http://www.elsevier.nl/locate/entcs/volume6.html.

[7] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Math. Soc. Lecture Note Series 64 (Cambridge University Press, 1982).

[8] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, Rendiconti del Seminario Matematico e Fisico di Milano 43 (1973) 135–166.

[9] F.W. Lawvere, Taking categories seriously, Revista Colombiana de Matemáticas XX (1986) 147–178.

[10] J.D. Lawson, Spaces of maximal points, Math. Structures Comput. Sci. 7 (1997) 543–555.

[11] J.J.M.M. Rutten, Elements of generalized ultrametric domain theory, Theoret. Comput. Sci. 170 (1996) 349–381.

[12] M.B. Smyth, Totally bounded spaces and compact ordered spaces as domains of computation, in: G.M. Reed, A.W. Roscoe and R.F. Wachter, eds., Topology and Category Theory in Computer Science (Oxford University Press, 1991) 207–229.

[13] K.R. Wagner, Liminf convergence in ω-categories, Technical Report, Cambridge University (1995); also: Theoretical Computer Science 184 (1997) 61–104.

[14] K.R. Wagner, Some well-known facts about adjoint bimodules, Preprint, 1995.

[15] K. Weihrauch and U. Schreiber, Embedding metric spaces into cpo's, Theoret. Comput. Sci. 16 (1) (1981) 5–24.